THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

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THE MATHEMATICS TEACHER

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ON THE CURRICULUM OF MATHEMATICS.

By Isaac J. Schwatt.

(*Continued from p. 177.*)

The student must thoroughly understand the meaning and the philosophy, so to speak, of each mathematical concept presented to him. It takes a long time before he familiarizes himself so thoroughly with the conceptions of any mathematical subject so that he gets their significance, meaning and spirit, and the ability and facility to apply them readily. Few students have a clear understanding of the quantitative meaning and significance of the theorems of proportion, such as: "A line drawn parallel to a side of a triangle divides the other sides proportionally," or, "Similar triangles are to each other as the squares of the homologous sides." In all my experience I have not received from a pupil a satisfactory explanation of the truth that one divided by infinity is equal to zero, a conception used in secondary mathematics. The same is true of college mathematics. Few students who have studied Analytical Geometry are able to give the true meaning, for instance, of the equation of the straight line, $y=sx+m$, *i. e.*, that the ordinate of any point of the straight line is m greater than s times the abscissa of the point. These instances may be multiplied to quite an extent.

In arithmetic the pupil, as a rule, is unable to understand the reasons for every step taken. He performs most of the

operations mechanically. When he is more mature the fundamental operations must again be taken up with the view of gaining clear ideas of the principles on which they are based. As I have already stated, the acquiring of knowledge and of clear ideas is a matter of evolution. I doubt whether we ever gain at once a perfectly clear understanding of an idea, especially if it is a little difficult. It has to grow on us, so to speak. The oftener we reflect and the more we meditate upon it, the clearer it will become to us.

Every idea presented to the student must be adapted to the state of his mental development. The amount of subject matter covered must be such as to make it possible for him without overtaxing his strength to acquire a thorough knowledge and a perfectly clear understanding of every point of it. There must be a fair distribution of the work done in each of the classes in the school and college. The work must be graded in such a manner that each succeeding year the amount covered is a little larger, or the subject taught a little more difficult than in the preceding year. Neither teacher nor pupil should have reason to state that a certain term, unless it is the last term, is the hardest throughout the entire course, or the easiest, unless it is the first term.

The student must not be permitted to do any work by empiric rules. There is, for instance, a pernicious practice of solving quadratic equations by a formula. This mode of effecting the solution does not take any less time than by logical steps. If the student forgets the formula and is not able to derive it, he is helpless. Nothing should be taught in the school or college which savors of the mechanical.

Some mathematical subjects are taught in the high school with little regard to the mental training which they are capable of yielding. Take, for instance, the subject of trigonometry. Most of the instruction in this subject consists in the solution of triangles which involve operations with logarithms. These the student performs, as a rule, in mechanical manner already induced by the use of the logarithmic table. The only occupation in which the trigonometrical solution of triangles may be necessary are surveying and astronomy. It is not to be denied that logarithmic computation develops the

qualities of neatness, of being orderly. But these qualities are much more common than the quality which it is the principal purpose of the school to develop, namely, power of mind. It is comparatively easy to secure the services of people who are able to do routine work with neatness and accuracy. But it is often difficult to find a person to fill a position in which mental power, power of discrimination and other qualities are necessary and indispensable.

There ought to be a difference in the kind of mathematics taught those who study the subject for cultural purposes only, and those who have to apply it. The latter must acquire such facility in the operations of algebra and trigonometry as will enable them to perform with ease the operations of calculus, a facility which can be gained only by long and continued practice. While such facility is indispensable to those who may have to use mathematics as a tool, it is not necessary for those who study mathematics for cultural purposes only. The cultural benefit from working examples is not as great as that obtained from a thorough understanding of principles.

The ideas of infinitesimal calculus are of great cultural value. In my opinion, every one who has received a higher education should be familiar with this powerful instrument of the mind. The study of calculus ought to be obligatory in any liberal course. If but one half of the year of two hours a week can be given to it and the work is thoroughly done, it will be sufficient to give the student an idea of the subject and add to his mental powers. I have often been asked by intelligent people whether I could give them in a few words an understanding of what calculus means.

During the time allotted to the study of the mathematical subjects as ordinarily given in the secondary schools, it is not possible for the average student to gain perfectly clear ideas of each of the topics taught; it is best therefore to confine the student to a few topics which he will need in the study of calculus. Very little plane geometry is necessary in the study of the higher branches of mathematics. From trigonometry the student must have perfect facility in the reduction of trigonometrical expressions. The solution of triangles and trigonometrical equations which are performed in a more or less

mechanical manner are not essential. Of analytical geometry only the equations of the conics and an ability to plot the curve corresponding to a given function are required. A complete knowledge of the graph of a function can only be obtained by means of calculus.

The mathematics which the average engineer uses in the practice of his profession, as a rule, is very simple. But if the engineer has to do creative work, or to carry out a project which is outside of the ordinary, he is in need of a great deal of mathematical knowledge and skill, sometimes a knowledge of the most advanced branches of the subject. This he cannot obtain during the short four years of his college course. The engineer will have to specialize in such mathematical work just as he must specialize for the more difficult engineering problems. In this respect the medical schools which are the oldest professional schools are best organized. The future physician while at school is given the fundamental principles of his science. If he is to pursue a specialty, he must supplement his knowledge after graduation. No physician without specializing will attempt to treat the diseases of organs like the eye or the ear, or of any complicated disease, for that matter.

To do creative work in mathematics requires, besides a thorough familiarity with the subject, peculiar abilities and powers, gifts with which only a few are endowed. This is true of the elementary as well as the advanced subjects. I fear that mathematicians have not sufficiently emphasized this fact. One may be thoroughly familiar with all the theorems necessary to perform a construction in plane geometry, and yet may not have the peculiar ability to combine these in such a manner as to effect the solution. But the average student ought to be able to follow and comprehend the fundamental ideas of the mathematical disciplines and derive the benefit for his mind which the knowledge of these ideas is capable of yielding.

I see, however, no reason why the study of numbers could not be used as a principal tool, as far as the instruction in mathematics is concerned, for developing power of mind, especially with those who will not continue their studies beyond the high school. We sometimes unconsciously attribute to a person who has studied geometry and trigonometry a broader and better

education than to one whose knowledge only extends to operations with number. But a knowledge of some of the properties of the triangle and of the relations of the trigonometric functions has no more cultural value than that of the properties of decimal fractions and periodic decimals, and the different tests of divisibility of number, etc. It is better that the number of tools be few but used with skill. It is not the number of books which a person has read that counts, nor the number of subjects which he has studied at school, but the mental benefit which he has derived from that study.

A great deal of mental training may be derived from solving some of the problems ordinarily given in algebra, by arithmetical processes. It is an invaluable exercise for developing power of mind. Some of the operations of algebra, like the simplification and reductions of fractions and surds are important exercises for training in accuracy, in bringing before the mind rules and principles previously learned and applying them instantaneously to the example on hand, adhering to a strict sequence and order of operations. But there is a peculiar benefit to be derived for the mind from solving problems by means of arithmetical processes. The solution of these by means of equations contains little educational value, while their solution by arithmetical processes often requires a great deal of concentration of mind. Take for instance problems like the following: "I have in mind a number of six digits, the last one on the left being one. If I bring this digit to the first place on the right, I shall obtain a number which is three times the number I have in mind. What is the number?" Or: "A grocer has two kinds of coffee, one at 64 cents and the other at 40 cents per pound. How much of each kind must he take to make a mixture worth 55 cents per pound?" Their arithmetical solution is an excellent exercise for the mind.

I know how difficult it is to break with tradition; I know that it is easier to agree and to follow the trodden path than to disagree and break loose from established and well-rooted practices. But progress necessitates a continuous change and an adaptation to changing conditions. Education seems to have met the change by extending the scope of the curriculum and thereby weakening instead of strengthening its effect.

The best material and the most perfect tools in the hands of the unskilled are of little avail. But the skilful even with inferior material and tools will show their mastery. The teacher with high qualifications and strong personality will help even the least able among his students. The teacher must be a thinker if he is to stimulate his pupils to think. He must be safe and sane, and wise and just, if he is to mould and frame the character of the rising generation. The teacher must be constantly improving himself if he is to give to his pupils a thirst for knowledge and a desire for self-improvement.

It is the boast of modern industry that machinery has been perfected to such a degree as to make it possible in some cases to dispense with intelligent guidance by man. But all the modern physical equipment used in education, which, if properly applied, is a great help in accomplishing its purpose, cannot replace the able teacher, nor can the most excellent textbook take the place of the living word.

No child should be during his entire school life under the influence and guidance of a man or woman only. The time is past when the mother alone had to do with the bringing up of the children and the father was considered only the bread winner. It was true then that: "A wise son maketh a glad father; but a foolish son is the heaviness of his mother" (Proverbs 10, 1). It is the coöperation of the parents, both father and mother, which gives a harmonious development to the character of the child. The good influence of both father and mother upon the child is bound to produce a more perfect man or woman, each maintaining the principal qualities of his or her sex. The man will be better fitted for the struggle in life and for his intercourse with both men and women, if he has acquired some of the gentle qualities of his mother and his woman teacher. The same is true of the girl, who in these days cannot lead the secluded life of former days within the walls of the house. She also will be better fitted for life if she has acquired some of the faculties which are characteristic of men.

As I have said in another place, the teacher enjoys more leisure than the followers of any other profession. There is no other occupation or profession in which even the beginner is free from the actual performance of his duties for a period

ranging from one fourth to one third of the year. This estimate includes also Saturdays, on which day, as a rule, no sessions are held. To whom much is given, of him much will be expected. It is therefore the duty of the teacher to use his leisure properly, not only in making himself proficient in the studies which he has to impart to his pupils, but also in informing himself of the world and the things in it, that he may be well fitted to influence his pupils for all that is right, noble and just. By example, as well as precept, the teacher must be to the student an inspiration and a stimulus to both moral and mental superiority.

Besides that many persons have assumed the responsibility of parenthood without being well fitted for this great task, and inasmuch that the child spends the greater part of the day under the influence of the teacher, or in preparing his lessons; the teacher therefore, should be especially well fitted to influence him toward all that is good and noble. He ought to be a specialist so to speak, in the two principal functions of education, the moulding of character and the development of the mind. In moral questions and in questions of health, not the majority but the minority, and very often the smallest minority, determines the status of a community and of the world at large. The unintelligent or ill-advised action of one individual may bring misfortune to, and affect the happiness of many.

The development of the moral and intellectual powers of the individual should be the principal aim of education. The school should inculcate in the young right habits, habits of honesty, of sincerity, of justice, of thoroughness, of striving for perfection in all things. These are, after all, a matter of habit. If we acquire the habit of being thorough, of being orderly, of doing things well, we will not be satisfied unless, whatever we do, we do it right. We shall find it is more difficult to break the habit than it was to acquire it. We must inculcate in the young the habit of right action while they are under our immediate influence and control.

We hear a great deal of the necessity for carrying on campaigns of education to enlighten the public on questions relating to their health, to their morals and their comfort, that they may have regard for the interest of others, not only for this

generation, but for all generations to come. The latest expression of this is found in the current agitation for the conservation of natural resources, etc. If we educate the young to have a receptive mind, a willing heart, an open ear, and a susceptible intellect, future generations will have less need for educating the public at large. It is difficult to get a hearing from those who are no longer responsible to us. The most instructive teaching and preaching I have heard was attended, as a rule, by such as needed it the least. Those who should have listened were not in the audience.

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Let me but do my work from day to day,
In field or forest, at desk or loom,
In roaring market place or tranquil room,
Let me but find it in my heart to say,
When vagrant wishes beckon me astray,
This is my work, my blessing, not my doom.

Of all who live, I am the one by whom
This work can best be done in my own way.

Then shall I see it not too great or small
To suit my spirit and to prove my powers;
Then shall I cheerfully greet the laboring hours,
And cheerfully turn, when the long shadows fall,
At eventide, to play, to love, and rest,
Because I know for me my work is best.

—*H. Van Dyke.*

FORMAL DISCIPLINE.

By W. H. Metzler.

I think the student of history would find few, if any, exceptions to the rule that what vitally touches everyday life and has existed for long years has considerable truth underlying it. New things will come in and give new light and show up old things in their true light, but they are not likely to entirely upset beliefs of long standing.

The belief in formal discipline has existed for a long time and it passed unchallenged until Herbart and his followers denied its right, though protests had been made earlier by Rousseau, Pestalozzi and others. The logical conclusion from this doctrine—that discipline obtained from the study of one subject was largely if not wholly useful in the study of another—was thought to be that a person should be almost equally good in everything and especially in those subjects most alike. It being patent to all that people are not equally good in everything the conclusion followed that the doctrine was wrong. At first the discussion was all from the deductive side, but of late experiments have added much to establish a true theory.

While the older defenders claimed too much, it seems equally true that some, at least, of the opponents denied too much. The swing from the one position to the other went too far, but the pendulum is undoubtedly settling down and finding its true amplitude.

The theory of formal discipline, or perhaps as it might better be called, general discipline in its older form seems to have been based upon a conception of the mind as an aggregate or a collection of faculties any one of which when disciplined by the study of one subject could use the strength thus gained in the study of any other. There seems now to be good reason for considering the functions of the mind as interrelated and interdependent. We cannot attend to different things equally well at the same time.

This problem of formal discipline has been taken up recently with renewed vigor both from the deductive and inductive sides.

Some experiments have been made which, according to the authors, showed that the mind has no general functions or transferable powers, but that all is specific. It is, however, quite certain that they did not take into consideration all the facts involved, and hence were mistaken in their conclusions. It is unfortunate that some men fresh with the enthusiasm of a new idea, or with too great a zeal to overthrow an old one, have drawn hasty conclusions from ill-conceived and poorly digested experiments. A writer recently made the following statement: "As there is no exact correspondence between mathematical and life situations, a good mathematician may be a poor reasoner in nonmathematical situations in life, and a good reasoner in such situations may be a poor reasoner in mathematics. These statements are not opinions but experimentally demonstrated facts." * While there may be no exact correspondence between mathematical and life situations there is, however, a correspondence and if properly taught there is no reason why those proficient in mathematics should not be able to handle problems of life better on account of their mathematical training.

The experiments themselves would seem to require little attention if such sweeping statements had not been made with regard to them. The first experiment consists in giving to high school students the following two sets of questions:

In Geometrical Reasoning.

1. Prove that the bisectors of the interior angles of a trapezoid form a quadrilateral, two of whose angles are right angles.
2. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base of the triangle.
3. If the bisectors of the equal angles of an isosceles triangle meet the opposite sides in D and E, prove DE parallel to the base of the triangle.

In Practical Reasoning.

1. Give all the reasons you can why a high school education is a good thing.
2. Why should the town rather than the parents pay for the education of its children?

* Cf. Lewis, *School Review,* April, 1905. Horne, *Education,* May, 1909.

3. Which is of more value, physical or mental training? Give all the reasons you can for the position you take.

The results of this test showed, of course, that those who answered the questions in geometry well did not answer the others best. In my judgment the test was valueless for the following reasons:

1. The test in geometry was too narrow, depending upon too few principles, which those who answered well might have memorized.

2. To reason about anything we must know the data on which the reasoning is based, but students of high school age would not as a general thing know the data of such educational questions and therefore could not reason about them. Their answers would be largely what they had heard others say.

3. Both tests depend too much on chance information and not on reasoning.

The second test was to take those college students who were good in mathematics and see if they were good in certain law courses which require a good deal of reasoning. Here again the ability to reason depends upon their knowledge of the facts and there was nothing to show these were the same for all.

Other experiments such as those of Coover and Angell * on sense-discrimination, of Ebert and Meumann,† of Wench,‡ and of Fracker § on memory are quite conclusive in showing that there is a general gain in the case of the trained over the untrained in going from one subject to another.

The expermients of Ebert and Meumann show that there was not only a gain in going from one subject to another but that the gain was greatest in material that was most closely related to that practised upon. The gain was not only in quickness but in retentiveness and it persisted, as after a considerable lapse of time there was no loss in training but in some cases there was an actual increase in memory capacity.

The experiments of Roediger ‖ show that neatness made conscious as an aim or ideal in one subject will cause the student to be neat in other subjects.

* *Am. Jour. Psy.*, 1907.

† *Arch. f. d. gesam. Psy.*, Vol. IV., 1905.

‡ *British Jour. Psy.*, Vol. II.

§ *Univ. of Iowa Studies in Psy.*, June, 1908.

‖ *Ed. Review*, November, 1908.

Urbantschish found that sound stimulus increased the sensitivity of the subject for visual, gustatory, olfactory, and tactile stimuli. Epstine found that sound stimulus increased the visual stimuli. Vogt showed that habituation to distraction in one situation could be carried over to other fields.*

Observation shows that logical or rational memory grows with use and it would seem that this gain was to a considerable extent due to the improvements of associating ideas and, as it were, tying them up together. The more ideas we have the more points of attachment for new material so that learning carries with it an increased capacity for learning other things related to it. Besides there is a gain in habits of learning. A great gain to be derived from any study is the learning of good methods of work—to learn what the tools are and how to use them. This has a general or transferable phase as well as a particular phase. Problems in arithmetic that seemed hard in school appear quite easy in later life though the individual may not have thought of them in the meantime and may not have been studying closely related subjects during the interval.

Some experiments would seem to show that there was a transfer of reciprocal interference as well as a transfer of improvement and that a person may become immune to improvement by practise. This last is probably due to habits becoming fixed and less elastic after long training.

It seems therefore fairly well agreed now that there is a transfer of discipline from one subject to another and the only question is the extent of the transfer and how it takes place. It is thought by some that this transfer of gain was due to "identical elements" in the subjects. The elements may be identical as to substance, as to method, or as to aim.

The conscientious effortful pursuit of any study results in rendering the mind more efficient in other lines of study. There is no development without conscious purposeful effort. Where there is little effort there is little development. In connection with the question as to whether one subject is better than another for developing these powers it should be noted that perhaps no subject in itself teaches anything but is the means or occasion for bringing out the lessons. It would seem, how-

* Cf. Stephen S. Colvin, "Some Facts in Partial Justification of the So-Called Dogma of Formal Discipline," *University of Illinois Bulletin.*

ever, that those subjects which in their study require most effort to acquire any particular power were best suited for developing that power.

A very important gain from any study properly pursued is an insight into the methods and technique of thinking. Another gain is the creation of ideals such as accuracy, neatness, system, etc.

Every experience has in it the possibility of generalization, but its value depends upon the way it is worked out by the individual. Another gain, therefore, in the proper study of a subject is the habit of making generalizations. Pupils should be taught to look all around a subject and see its bearings, but this cannot be done by a narrow teacher. On the other hand a narrow teacher can do much to prevent a student from getting much general through special discipline, and from transferring his training and culture from one subject to another. Pupils should become interested seekers after truth and not dogmatists.

Thinking has its origin in some perplexity or doubt, then past experience or knowledge comes in to offer suggestions as to a solution. These suggestions must be critically examined and judgment suspended during further inquiry. Good thinking then, involves a fund of experiences, facts, and principles from which suggestions proceed, a fertility of suggestion, and an appropriateness and consistency in what is suggested. A large experience is not so necessary as a trained experience. It is not information that makes a man educated, but rather well grounded attitudes and habits of discriminating tested beliefs from mere assertions and opinions. Professor Dewey says "The only information which, otherwise than by accident, can be put to logical use is that acquired in the course of thinking." *

With some people suggestions come readily while with others they come slowly and we call them bright or dull accordingly. There is also a difference in the number or range of the suggestions, but the best mental habits require a proper balance between too many and too few. Again suggestions may vary as to their quality. Slowness of suggestion is not always to be deplored. Better slowness with depth than readiness with shallowness. Finally suggestions must be appropriate and the mind held to an orderly course. The serious vocations of adults

* Dewey, "How We Think," p. 53.

tend to make them orderly in their thinking, but modes of activity must be found to accomplish this same object for the student.

In all reflection there is a double movement, first from the original facts or data to the suggested conclusion—the inductive movement, and then from this general conclusion back to the particular facts—the deductive movement. The inductive movement is towards the discovery of a general principle and the deductive towards its testing. Each process throws light upon the other and students should not be allowed to make an induction, *i. e.*, draw an inference without following it up deductively to see if they understand it. He must be able to apply his general principles to particular cases and to new situations. The deductive phase is quite as important as the inductive. The two should be combined to produce critical thinking. "Only deduction brings out and emphasizes consecutive relationships, and only when relationships are held in view does learning become more than a miscellaneous scrap-bag." *

These are the rudiments of mental discipline and it is the business of education to supply the conditions for their cultivation. The aptitude of acquiring them varies with the individual but they are not natural endowments. We are endowed with power of thought, which education does not create but trains to think well.

There is of course danger in separating logical thought too much from every day affairs so that students cannot apply it, just as there is danger of making some studies too mechanical as reading sometimes is, or of making others merely informational as geography sometimes is.

Because in the training of some people logical thought has been made too much an abstract thing in itself and they have not been able to apply it to practical affairs it has been thought by a few that the study of those subjects which are most abstract tend to produce this condition. Such fault lies with the teaching and study of the subject and not with the subject itself.

There is another danger which consists in putting too much stress on the product and too little on the process of attaining

* Dewey, *ibid.*, p. 97.

the result. A student may easily come to think that the answer is the all important thing. The methods of the teacher may be responsible for this, for method covers not only what we consciously employ for the purpose of mental training, but what is done without conscious effort which reacts upon the activity and mental attitude of the pupil. There is too much testing by observable results and by these alone. Too few questions are asked and answers read with a view of getting at mental habits and powers. Because mathematics lends itself more readily than other subjects to the exact measurement of results, it is quite possible that we as teachers may look too much to the results of the information acquired and too little to the results of the processes and habits developed. We ask A and B to prove the same proposition in geometry giving them 100 and 75 respectively for their answers, but are we at all sure that these numbers measure their relative knowledge of the proposition? Perhaps in any particular case we cannot do better but we can keep more constantly in mind the possibility of error such markings may contain.

It is thought by some that things must have an interest in themselves to be of value and we are asked to pay more attention to utilitarian and practical aims and to replace the more abstract subjects of the curriculum by those of greater interest. Before doing this, however, it might be well to pause long enough to ask what is meant by practical and utilitarian subjects and whether the interest demanded is the mere entertainment type or something more lasting. As Professor Dewey well says it is "not the thing done but the quality of mind that goes into the doing that settles what is utilitarian and what is unconstrained and educative." * He also says: "Truly practical men give their minds free play about a subject without asking too closely at every point for the advantage to be gained; exclusive preoccupation with matters of use and application so narrows the horizon as in the long run to defeat itself. It does not pay to tether one's thoughts to the post of use with too short a rope. Power of action requires some largeness and imaginativeness of vision. Men must at least have enough interest in thinking for the sake of thinking to escape the limits of routine and custom. Interest in knowledge for the sake of knowledge,

* Dewey, *ibid.*, p. 167.

in thinking for the sake of the free play of thought, is necessary then to the emancipation of practical life—to make it rich and progressive."*

Interest is a good thing where and in so far as it is possible, but some things are not in themselves interesting and any interest in them must be acquired. There must be effort to attend to or to concentrate on some things and a large part of the discipline derived consists in that attitude of mind which tolerates unpleasant and distracting things. Spencer defined the educated being as one who did what he ought when he ought whether he wanted to or not.

Some teachers succeed in arousing interest and enthusiasm, in communicating information, and in developing energy, but fail in developing habits of attending to details that makes certain mastery over means of execution, which is power. Other teachers succeed in developing skill, facility and mastery of technique, but fail in developing power of discrimination, a sense for ideas, and enlarged mental vision.

To study mathematics for its applications is to aim at quantity rather than at quality. The better grade of students will get some discipline in connection with the facts but the majority will fail to get those unique results in training and culture which the study should supply. The ideals of accuracy and truth will be lacking. They will not realize that that which is but partially true is of no value in human experience and will not appreciate the ideal of truth—self sufficient truth—for truth's sake. Truth is one of the greatest passions of the human race, it is fundamental and should never be denied the possibility and opportunity of exercising its uplifting influence towards the ideal. There is so much in the world that tends to keep us down on the lower plains of life that we as educators should cherish the opportunity and means that will not only give us higher ideals but which will draw us up with a fixed purpose to live nearer to them.

Another thing of which we as teachers must be careful is our attitude towards problems of conduct, because the way they are met has an influence which extends to every other mental attitude. If we are not in the habit of giving serious inquiry and reflection to vital questions of conduct it is not at all likely

* Dewey, *ibid.*, p. 139.

our habits of thought will have much influence in less important matters.

To secure this general training it seems fairly well established that we must observe the following general rules:

1. Make the activity, such as neatness, etc., which we wish to transfer, an object of thought. The habit and its general significance and bearing must be understood by those who are to acquire it.

2. General attitudes of feeling and will should be cultivated through specific training. Respect for authority, etc., are qualities readily transferable.

3. The general art of learning should be acquired through specific study. How to work with the highest effectiveness, with the greatest economy of effort and time is an accomplishment too seldom attained, and one which receives too little attention.

4. Such habits as observation should be cultivated in as wide a field as possible.*

The great business of the schools is to teach pupils how to think well, to have proper ideals and methods and to have a will sufficiently developed to make them do what they ought.

We have seen that the most important thing that a student should obtain from the study of a subject is not information but mental power, proper ideals and ambitions, and these are for the most part transferable from one subject to another. Our duty then as teachers of mathematics is plainly to ascertain just what may be expected from the study of each of our subjects and aim to accomplish that. What each subject is good for is a study in itself and has been treated elsewhere.

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* Cf. Stephen S. Colvin, "Some Facts in Partial Justification of the So-Called Dogma of Formal Discipline," *University of Illinois Bulletin*.

A SHORTENED FORM OF SYNTHETIC DIVISION AND SOME OF ITS APPLICATIONS.

By Eugene R. Smith.

The ordinary form of synthetic division is probably familiar to most if not to all teachers of mathematics. I think the best arrangement for division by a binomial divisor is as follows: To divide $2x^3 - 5x^2 + 3x - 7$ by $x - 2$,

$$\begin{array}{r|l} & 2 - 5 + 3 - 7 \\ 2 & \quad\;\; 4 - 2 + 2 \\ \hline 1 & 2 - 1 + 1 \;|\; - 5 \end{array}$$

Giving a quotient $2x^2 - x + 1$ and a remainder -5.

This method of division can equally well be applied to any function of one or two variables, as, $6x^4 - 7x^3y + 5x^2y^2 + 7xy^3 - 11y^4$ divided by $3x^2 - 2xy + 3y^2$

$$\begin{array}{r|l} & 6 - 7 + 5 + 7 - 11 \\ -3 & \qquad\; - 6 + 3 + \;\;3 \\ 2 & \quad\;\; 4 - 2 - 2 \\ \hline 3 & 2 - 1 - 1 \;\;|\; 8 - \;\;8 \end{array}$$

Where the quotient is $2x^2 - xy - y^2$ and the remainder is $8xy^3 - 8y^4$

I might add that I believe in teaching this method of division early in first year algebra, and encouraging its use throughout the course. It will accomplish the necessary division with a minimum of labor and a maximum of speed, neatness of arrangement, and economy of space. The division by a binomial is of the greatest importance, and that form can be still farther shortened without any great increase in difficulty

$$\begin{array}{r|l} & 2 - 5 + 3 - 7 \\ 2 & 2 - 1 + 1 \;|\; - 5 \end{array}$$

the work being as follows: Twice 2 and -5 equals -1, twice -1 and 3 equals $+1$, twice 1 and -7 equals -5.

It is this method and some of its applications of which I wish to speak. One of its best uses is in finding plotting values. I will use a cubic equation, but the principal applies equally for

any degree from a quadratic up, if but one variable is involved.

To plot $x^3 - 4x^2 - 2x + 8 = y$

x		y	
-2	$1 - 6 + 10$	-12	
			$\rightarrow$ root
-1	$1 - 5 + 3$	$+5$	
0	$1 - 4 - 2$	$+8$	.......(original equation)
1	$1 - 3 - 5$	$+3$	
			$\rightarrow$ root
2	$1 - 2 - 6$	-4	
3	$1 - 1 - 5$	-7	
4	$1 \quad 0 - 2$	0	root; depressed equation $x^2 - 2 = 0$

Note that the x and y columns give the plotting values, that a change in sign in the y column shows between what two integers (in the x column) the root lies, and that a zero in the y column shows a root (in the x column), the part between the two columns being the depressed equation, from which in this case the exact values of the other roots can be found. It is evident that this arrangement can with equal value be used in drawing the graph of any function of one variable, or in advanced algebra in solving (for either commensurable or incommensurable roots) an equation of higher degree than the second. Even if Sturm's Theorem were to be used later, this would be the best way to start, and if the location is to be done without Sturm's, there can be no question of its convenience. The method works equally well with fractions, so gives a very simple way to substitute a fractional value.

Again, suppose the roots of an equation are to be diminished by some number, as the roots of $x^3 - 4x^2 - 2x + 8 = 0$ by 2. The work could be arranged

$$\begin{array}{c|llll} & 1 & -4 & -2 & +8 \\ 2 & 1 & -2 & -6 & \lvert -4 \\ & 1 & 0 & \lvert -6 & \\ \hline & 1 & +2 & & \end{array} \qquad \text{The new equation being } x^3 + 2x^2 - 6x - 4 = 0$$

This would, of course, be applied to Horner's method, and the work would be very much simplified. If a pupil is weak in arithmetic, he would have to drop the shortened form after one

or two decimal places, but I have had pupils who preferred to use it throughout by combining figure by figure, instead of term by term. In handling terms having different signs the Austrian method of subtraction would be used.

Still another application comes in Partial Fractions. If it is required to separate

$$\frac{3x^4 - 2x^2 + x - 1}{(x - 1)^5}$$

into partial fractions, dividing by $x - 1$, and keeping the successive remainders,

$$\begin{array}{r|l}
 & 3 \quad 0 - 2 + 1 - 1 \\
1 & 3 + 3 + 1 + 2 \;|\; + 1 \\
 & 3 + 6 + 7 \;|\; + 9 \\
 & 3 + 9 \;|\; + 16 \\
\hline
 & 3 + 12
\end{array}$$

the answer being

$$\frac{3}{x - 1} + \frac{12}{(x - 1)^2} + \frac{16}{(x - 1)^3} + \frac{9}{(x - 1)^4} + \frac{1}{(x - 1)^5}$$

This method is easily proved, and is, I believe, the shortest way yet found for this kind of example. The degree of the numerator has no effect on it.

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CHEERFULNESS.

If cheerfulness knocks at our door we should throw it wide open, for it never comes inopportunely; instead of that we often make scruples about letting it in. Cheerfulness is a direct and immediate gain—the very coin, as it were, of happiness, and not, like all else, merely a cheque upon the bank.—*Schopenhauer.*

A SIMPLIFICATION IN ELEMENTARY TRIGONOMETRY.

By W. H. Jackson.

It is generally admitted that consideration should be paid rather to the normal student who forms the bulk of the class than to those few who find the subject particularly easy or particularly hard.

The normal student who studies trigonometry learns numerous formulæ and forgets them very soon after his course has been completed. If on some later occasion he finds an opportunity for applying the knowledge which he once possessed, his recollections have probably merged into confused memories of endless formulæ and he finds it impossible to extricate that one for which he has particular need. To a certain extent this is inevitable; but there is one change in the mode of presentation usually adopted in the text-books which would do something to lessen this difficulty. If a single method can be made to replace a number of processes there is, firstly, less to remember and, secondly, more practice in using what is retained and a double gain is made.

The method which it is the object of this article to advocate has reference in particular to the relations between the trigonometrical ratios of a single angle and their use in the proofs of identities, the solution of equations, and the reduction of complicated trigonometrical expressions to their simplest form. Of these three classes of problems the last is at once of the greatest practical importance and that one in which the method is of the greatest use.

The method is to translate each trigonometrical problem into an algebraic one.

(i) Select that one of the six ratios which occurs oftenest and denote it by x.

(ii) Construct a right-angled triangle in which the sides forming the numerator and denominator of the ratio are of lengths x and 1 respectively and by means of the theorem of Pythagoras find the length of the third side.

(iii) From the figure read off the values of any of the other five ratios which are required and substitute in the expression considered.

The following examples will suffice to illustrate the method.

(*a*) Prove that

$$\sin A\ (\cot A + 2)(2 \cot A + 1) = 2 \csc A + 5 \cos A.$$

(i) Write $\qquad \cot A = x = \text{adj./opp.},$

(ii) and $\qquad \text{hyp.} = \sqrt{1 + x^2}.$

(iii) The equation is therefore equivalent to

$$\frac{1}{\sqrt{1+x^2}}(x+2)(2x+1) = 2\sqrt{1+x^2} + \frac{5x}{\sqrt{1+x^2}}$$

or

$$\frac{(x+2)(2x+1)}{\sqrt{1+x^2}} = \frac{2(1+x^2)+5x}{\sqrt{1+x^2}},$$

which is at once seen to be an identity.

(*b*) Simplify

$$\csc^2 A \tan A - 2 \sin A \cos A - \sin^2 A \tan A.$$

(i) Write $\qquad \sin A = x = \text{opp./hyp.},$

because though $\tan A$ occurs as often as $\sin A$, the reciprocal of the former, $\cot A$, is absent while the reciprocal of the latter, $\csc A$, is present.

(ii) Hence $\qquad \text{adj.} = \sqrt{1 - x^2}.$

(iii) Making these substitutions, we obtain

$$\frac{1}{x^2}\frac{x}{\sqrt{1-x^2}} - 2x\sqrt{1-x^2} - x^2\frac{x}{\sqrt{1-x^2}} = \frac{1 - 2x^2(1-x^2) - x^4}{x\sqrt{1-x^2}}$$

$$= \frac{1 - 2x^2 + x^4}{x\sqrt{1-x^2}} = \frac{(1-x^2)^{\frac{3}{2}}}{x} = \frac{\cos^3 A}{\sin A}.$$

Some of the first objections which naturally arise are as follows: (1) This method fails to give practice in using the

relations between the six ratios, (2) it is mechanical, (3) it is often clumsy.

In reply to which the following may be said: (1) It renders unnecessary the ordinary relations between the ratios only because it carries them back to their source in the definitions and the theorem of Pythagoras and to do this is a gain and no loss. (2) Economy of thought means replacing hard processes by simpler, more mechanical ones. The method is not mechanical in the sense that it hides or obscures the processes involved. (3) If we raise the treatment of identities from the uninstructive level of puzzle solving to the rank of sure and methodical solution a feeling of power is gained which may well balance an occasional clumsiness.

If the method is applied according to the preceding rules to the corresponding problems involving two or more angles it is undoubtedly clumsy; but a firm foundation has been laid for the treatment of those questions also. It has become clear that the identity of two expressions can not be immediately perceived until they are ***both expressed in terms of the same units.***

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HAVERFORD, PA.

So long as we love, we serve; so long as we are loved by others I would almost say we are indispensable: and no man is useless while he has a friend.—*Robert Louis Stevenson.*

SOLID GEOMETRY.

By Howard F. Hart.

If I should make a culture-epoch-theory investigation into my own experience with a certain theorem I would probably put my experience into three stages as follows:

1. "If the unit of measure for angles is the right angle, the area of a spherical triangle is equal to its spherical excess multiplied by the area of a tri-rectangular spherical triangle."

2. "A spherical triangle equals a lune whose angle is half the spherical excess of the triangle."

(3) $$T_{A,\,B,\,C} = L_{\frac{A+B+C-180°}{2}}.$$

Neglecting the differences in form between the first two due to the fact that the first is a theorem of mensuration while the second is a theorem of geometry, we see that the second is a simpler statement than the first of the relation between the surface of a triangle and that of the sphere. The third is an accurate, simpler, symbolized form of the second. In it certainly the proposition has been reduced to its lowest terms.

The device, then, consists in using a notation in which the lune of angle A is denoted by L_A; the spherical triangle of angles A, B and C by $T_{A,\,B,\,C}$; the spherical polygon of angles $A_1A_2 \cdots A_n$ by $P_{A_1A_2\ldots A_n}$; the spherical wedge of angle A by W_A, etc.

In this notation one group of theorems of spherical geometry is stated as follows:

(1) $$L_A \pm L_B = L_{A \pm B},$$

(2) $$1/n \cdot L_A = L_{A/n},$$

(3) $$(L_A \cong L_B)_{A=B},$$

(4) $$\frac{L_A}{L_B} = \frac{A}{B},$$

(5) $$L_{2\pi} = 4\pi R^2,$$

(6) $$P_{A_1A_2\ldots A_n} = L_{\frac{(A_1+A_2+\ldots+A_n)-(n-2)180°}{2}},$$

Another group might be also given as follows:

(1) $$\frac{W_A}{W_B} = \frac{A}{B}, \quad W_A \pm W_B = W_{A \pm B}, \quad 1/n \cdot W_A = W_{A/n},$$

(2) $$(W_A \simeq W_B)_{A=B},$$

(3) $$W_{2\pi} = \tfrac{4}{3}\pi R^3.$$

If also $T_{A,B,C}$ and $T'_{A',B',C'}$ denote polar spherical triangles of sides a, b, c and a', b', c' the same idea can be extended to that group of propositions, viz., "If T and T' are polar then $a =$, in measure, $180° - A$," etc.

I first used the notation in simplifying the proof of the proposition quoted at the beginning of this paper and then with the area group and now wherever it is worth while.

Its chief advantage is its power of simplification. It is that sort of advantage which an open-faced watch case has over an hunting case. The real meaning of a proposition is at once evident.

If you will observe the statement of theorems (1) and (2) in the first group you will see that in addition to the advantages of a symbolic expression we have also those due to an operative system (*e. g.*, the lunes add by adding their angles which fact is indicated by an operation in the symbols).

Suppose that we consider the first theorem to follow after (5) in the area group and prove it.

So. Given $T_{A,B,C}$.

To prove $T_{A,B,C} = L_{\frac{A+B+C-180°}{2}}$.

Proof 1. Extend the sides T to form L_A, L_B, L_C,

2. $\widehat{BC'} = \widehat{CB'}$, $\widehat{AB'} = \widehat{BA'}$, $\widehat{AC} = \widehat{A'C'}$ $\quad (=^s - =^s)$;

3. $AB'C = BC'A'$ $\quad$ (3 sides);

4. $L_{180°} = L_B + (L_A - T) + (L_C - T)$ $\quad$ whole = sum parts;

5. $2T_{A,B,C} = L_A + L_B + L_C - L_{180}$ $\quad (=^s \pm =^s)$;

6. $2T_{A,B,C} = L_{A+B+C-180°}$ $\quad$ cf. (1) above;

7. $T_{A,B,C} = L_{\frac{A+B+C-180}{2}}$ $\quad$ cf. (2) above.

The notation is helpful in solving exercises as the result is

found by merely substituting, *e. g.*, "Find the area of a spherical triangle whose angles are 70°, 80°, 90° on a sphere whose diameter is 20 in." (Hopkins.)

$$T = L_{\frac{70^\circ+80^\circ+90^\circ-180^\circ}{2}} = L_{30^\circ} = \tfrac{1}{12} L_{2\pi} = \frac{4\pi \cdot 100}{12} = \frac{100\pi}{3} \text{ sq. in.}$$

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CONTENTMENT.

Let not the thoughts of what is yet to come
  Mar the sweet blessings that are ours to-day;
How can we doubt, when every living thing
  Proves that His hand is leading all the way?

Ne'er yet a sorrow but has left the heart
  Stronger and nobler, having felt the pain;
Joy comes again to every troubled breast,
  Just as the sunshine follows after rain.

Drink in the sunshine, wear a happy smile;
  Brighten the way therein our steps have trod;
Do the kind act and say the cheering word;
  Lift up your heart in thankful prayer to God.

So shall the burdens of our life below
  Seem so much lighter and our hearts more strong,
If by our words, our acts, our prayers, our deeds
  We can make life one perfect, grand, sweet song.

SOME REMARKS ON APPROXIMATE COMPUTATION.

By M. J. Babb.

It has generally happened in the past that teachers of elementary mathematics have had little or no acquaintance with what is generally called the higher mathematics. Even when a boy went to college he promptly forgot his elementary mathematics, and if by chance he became a teacher in the secondary schools there was no one to advise with him on means of applying what he had learned to the work in hand. Necessarily he drifted into the stereotyped manner of teaching the subject and his college work was only a pleasant memory. The college teachers were often recruited from men who naturally had very little knowledge of the problems of elementary education. Very few would have been able to grasp the point of view of the youthful mind; many were thoroughly ingrossed in their subject and had found ideals of rigidity of statement and proof as far beyond the ken of the elementary schools as they were in advance of the notions of those who intuitively founded the subject wiser than they knew.

In this paper I am privileged to address a body of men and women who are banded together, not as college teachers, not as teachers of arithmetic or algebra or geometry, but as teachers of mathematics. The spirit of "put yourself in his place" is growing. The normal schools owe their very existence to the need of the elementary teacher for guidance and sympathy with his work not found in the college. There would have been no room for them had the colleges been interested in teaching how to teach boys and girls, much more scholarly work would have been done by the would-be teachers and instructors and sounder principles would no doubt have been evolved. The primary education in our country is excellent. The Mosely Commission claimed it far exceeded anything abroad and the development of the primary pupil here to be much in advance of anything they could offer. The plans and devices of the superintendents were pronounced to be marvelous in their completeness, but the

commission reported at home that the grammar and high school pupils here were behind the English pupil of the same age. This can only be explained in one way—our teachers did inferior work. In putting the elaborate schemes into execution they did not make them a part either of their own or the pupils' experience and while the result was showy, it was not lasting. It is not so much superintendents and principals and text books that we need but teachers who meditate and encourage meditation and originality. The inspiration of a great teacher is more than fine buildings or exquisitely worked out courses of study and machine-like plans for administering them. Let us have men and women of power for teachers. The good work of a teacher does not show immediately. The charlatan and the opportunist will attract more attention than the teacher. This is true even in the "professions" in which classification we are not yet admitted by common consent and as a result of our own weaknesses.

Much of the early mathematics consisted of a crude arithmetic for practical use on one side and pure puzzles for show purposes on the other. This tendency gave way to the idea that mathematics was always an exhaustive development of a particular subject for its own sake and its own method. There has been for some time a decided tendency toward unification, not merely a gathering together of the forces for a new series of separate exploits but a realization of the interrelations, one to one correspondences, even multiple isomorphisms, of subjects heretofore kept in watertight compartments hermetically sealed. We are met in such associations as this for mutual advice and assistance, and there is not one of us so wise that he cannot learn of the least of those among us.

It has been well said that the purposes of teaching arithmetic are the following: (1) Drill for accuracy and speed in computation. (2) Interest in nature. (3) Acquaintance with the mathematical type of thought. (4) Preparation for further study. If these are properly expressed they will have extension to the so-called higher branches of the subject.

We will probably agree that there are in general three stages in most problems in mathematics. First, a translation of the known existing data into mathematical symbolism more perspicuous to the eye. Second, manipulation of this symbolism,

according to laws fixed by reason or custom, keeping in mind all the time the one to one correspondence between the actual and our ideal problem. And lastly, a retranslation from the symbolism to ordinary language and a verification of the result with known fact.

It has been the neglect of the first and last of these stages and a partial neglect of the second that has made it possible for our ranks to be recruited at times from worn out preachers who have lost their voice and people not strong enough for other subjects. Some of our cotemporaries even now believe that any one can teach mathematics as every problem is either all right or all wrong and there can be but one answer and that one always in the book.

My uncle tells me that he once attended a dames school kept by two Quaker ladies of slender means, which slender means and the attitude in which the teaching profession was held were the sole reasons why they kept school. The highest mathematics taken was long division. He said he wrote down his divisor *out of the book*, then his dividend *out of the book*, then his quotient always *out of the book*. Then they made him multiply out that long tail below and why they made him do it he never knew, for they never told him and he believes they are placidly unconscious of it still.

I wonder how many of us as children wondered why in multiplication by a number of two or more digits we wrote the second set of figures one place to the left. I am sure it was long a puzzle to me, and my little boy came home from school the other day troubled over the same old question and was happy when I told him why.

Charles Dickens appreciated the radix method when he said in "Bleak House" that a certain man "was to his wife what zero is to nine in ninety—something with her and nothing without her." Place in mathematics as in the world is relative. The only real difference between the bank account of a teacher and John D. Rockefeller is where he has the power to place the decimal point.

Our ancestors like ourselves at a more or less remote date from the present spent many hours in earnest contemplation and admiration of their digits and finally evolved a method of enumeration which is now based on the original *ten* digits in our

ordinary operations with number and from which has developed the power series of more general or algebraic number. There is no virtue in the radix *ten*, to make a poor pun it just happened to be *at hand*. The number 392 is $3(10)^2 + 9(10) + 2$. Whether the whole accidental and intuitive system to which we have referred is unduly cumbersome and as my friend and constant source of inspiration, Dr. Schwatt, has intimated may be illuminated or replaced when we understand the relations that exist between certain varieties of incommensurable numbers such as π and ϵ and others doubtless belonging to the same or different systems is a question for further study. Note $\epsilon = 1 + 1/1 + 1/1\cdot2 + 1/1\cdot2\cdot3 + \ldots$ where the radix does not occur to a power but appears as a product of a sequence. Certainly our operations in mathematics will bear inspection and cannot in the spirit of the subject be taken entirely on faith.

We all hear it stated and the world in general believes that the mathematics is a sort of Dunderbach's sausage-making machine. An eminent professor of philosophy said recently that the advantage of the calculus was that all you had to do was to turn the crank and your answer came out ready to use. The modern psychologist inveighs against us in that our reasoning is in a particular narrow line and states that reasoning in a particular line will not necessarily benefit reasoning in general. The schools say that the best hours of the day and more than one fifth of the time during the first ten years of school life is spent on mathematics with very meagre practical results and little of culture value. Who is responsible for this attitude?

If I mistake not there is a feeling among teachers of mathematics that there is room for improvement in methods and sequence of subject matter.

Most of the literature at present is of the destructive, rather than constructive type. Much of it for a time must be inspirational and of a sort to encourage supplementing the present type of texts and the gradual omission of useless or obsolete topics. This work must necessarily be slow and must come from the teachers for no publisher would dare cut out of his book all of the useless and nonsensical ideas therein. Even if it were possible to write a perfect text there would still be the individuality of the teacher to consider. It is a well established fact that a fair system well understood and taught will lead to better results

than an excellent system in the hands of an unsympathetic instructor.

It is my purpose to discuss a certain type of mathematical thought because of its relations to practical work, accuracy and speed in computation, and as an aid to the understanding of the higher mathematics. I will make two statements:

1. *Every number we use in actual life is an average number.*
2. *The operations of mathematics will bear watching.*

Our judgments are based on the average of the sum total of our experiences. When we wish to measure any length accurately, we repeat the process several times and take the average of the several different estimates we have made as the most probable value. We can estimate a length to the nearest inch, or to the nearest sixteenth of an inch or to the nearest thousandth part of an inch, you measure to any degree of accuracy you may desire and I will obtain a closer approximation, and then you can come closer still with a more finely divided measure.

Here is the germ of the infinite series. The true length is the limit of the sum of this series. The difference between the true measure and your result is an infinitesimal. Your measurement is not exact. It is within the nearest artificial unit you may choose. The quantity you are measuring is not commensurable with your measure. Many a boy who is set to measuring a table as his first lesson in physical measurements presents a suspiciously uniform set of measurements and is sent back to do his work again. Many a surveying class goes back to the field to find that the angles of a triangle never sum up exactly to 180°. Now are any of these ideas impossible of comprehension to the average school boy. They are fundamental conceptions needed in higher mathematics and in practical work, and yet they are absolutely new to ninety-nine out of every hundred freshmen in college and engineering schools.

And now a word about the statement that the operations of mathematics will bear watching. Suppose we want the product of the two numbers 258.73 and 24.568 correct to five significant figures (note I did not say decimal places). See Problem 1. Placing units under units so as not to be troubled about place, we will multiply the numbers in the ordinary way except to change the order of the products as we do in algebra and obtain

as our product, 6356.47864. Now will it not occur to the thinking student that we have done an unnecessary amount of work in order to obtain five figures? Would not the process in Problem 2 have obtained the result more expeditiously?

Problem 1.	Problem 2.
258.73...	258.73...
24.568..	24.568..
5174.6....	5174.6....
1034.92...	1034.9....
129.365..	129.4....
15.5238.	15.5....
2.06984	2.1....
6356.47864	6356.5
6356.5	

The operation might be conducted in the following manner. Two tens by three hundredths gives six tenths, &c., for the first product. Now cross off the two tens that you have used in multiplying and you can do the same with the three hundredths for units times hundredths gives hundreths. You will however need to note that the product would have been four units by three hundredths gives twelve hundredths, which is nearest to one tenth and with the twenty-eight tenths next obtained make twenty-nine tenths and so on each time crossing off a figure from multiplier and multiplicand until you are forced to stop because you have no longer numbers of which to take the product. This process is evidently much shorter than the one in ordinary use for exact numbers. Now let us suppose these are not exact numbers but numbers such as we obtain from measurements, one of the numbers being given to the nearest hundredth of the unit and the other to the nearest thousandth part. (There are approximate integers as well.)

Now the eight in the thousandth place (Problem 1) is not quite correct neither is the three in the hundredths place, hence the four in the hundred thousandth place is even more doubtful, and since all the dots represent missing unknown digits our first problem is to add to a very doubtful number four other numbers of which we are entirely ignorant. Certainly the chances are nine to one against the sum being a number ending in four and as De Morgan says we have even greater doubt about some of the others. In fact we do not know very much if anything about

any unfinished column, and we certainly can place no dependence on any more of our work than we used in our abbreviated process (Problem 2). Even the last figure here will bear close scrutiny and we will find in general that the result is correct to one significant figure less than we obtained from our observation. In fact in long and complicated examples we had better correct to two less figures or make an examination of the limits of error which is easily done in any particular example. In this example you will notice that we are not absolutely sure of anything beyond the corrected tens place, and that our result is about an average of the possible values. The problem not only shortens up the work but calls attention to the accuracy of the result and prevents many errors.

PROBLEM 3.

```
               5.34831
   3.142) 16.80437921
          15 710
           1 0943
             9426
            15177
            12568
   2610     26099
   2514     25136
     96      9632
     94      9426
      2      2061
```

PROBLEM 4.

```
           5.348
   3.142) 16.80437921
          15 710
           1 094
             943
             151
             126
              25
              25
```

Note also the problem in division (see Problem 3) again supposing the numbers exact. It is convenient with young pupils to have the decimal point after the first figure of the divisor in order to make the division more conveniently but this is evidently not necessary. *We will note all through that there is absolutely no difference in the problem whether there is a decimal point expressed or understood. There should be no difference in the treatment of decimals and whole numbers.* The point merely changes the size of the unit. As soon as the remainder (921) is less than the divisor (3142), it is convenient to stop bringing down more figures and obtain the remaining digits of the quotient by continually contracting the divisor and dropping the rest of the dividend. Thus cutting off the 2 we see 314 is contained in the 2610 eight times, and since 8 times 2 is 16 we have

2 to "carry" from the cut off digit. Similarly cutting off successive digits the operation is as on the left.

The process is the same if only the dividend be approximate, but if the divisor is approximate the last figures in the dividend will be useless and we will be reasonably sure of only four figures in this case. There is evidently no need to have more digits in the divisor than in the quotient (see Problem 4). The extreme folly of annexing ciphers in such cases as this and in any case of supposing that a calculation of any kind could give results more accurate than our data or indeed as accurate except by accident is self evident. A certain celebrated chemist would not have formed an erroneous theory, nor would a prominent psychologist just recently have advanced an equally fallacious statement in one of our scientific journals if they had understood the operations they were so thoughtlessly using. I once asked one hundred public school teachers to multiply $\pi = 3.14159$ by the $\sqrt{2}$ and give result correct to two places of decimals and got one correct result. But what surprised me most was the fact that many elderly carpenters and machinists were able to do the problem. The carpenter in obtaining the square root of 2 would say it is more than 1. He would divide 1 into 2 and getting two would take the average $1\frac{1}{2}$, then he would divide $1\frac{1}{2}$ into 2 and get $1\frac{1}{3}$ and again take the average until he had the required accuracy.

It is absolutely impossible to write the $\sqrt{2}$ in our ordinary system of decimal notation and its conception requires an extension of our idea of the square root just as new to the ordinary pupil as the idea of fourth dimensional or n dimensional geometry is to a student of Euclidean geometry.

We can draw the diagonal of a square of unit side and thereby construct the geometric correspondent to our extended notion of the $\sqrt{2}$. But it is wise to keep in mind in such problems of correspondence that though a point on a line may be made to correspond to every real number the converse requires examination. I will here remark that it is probable that this idea of correspondence would save the secondary teacher much trouble in constructing graphs of quadratic equations on paper ruled in squares. For example, if the side of one square be taken as the unit, the diagonal is the $\sqrt{2}$. A rectangle, $\sqrt{2}$ by 1, has $\sqrt{3}$ for its diagonal. The $\sqrt{5}$ is obtained from the diagonal of a

rectangle, two units by one unit. For the $\sqrt{6}$ we may take the diagonal of the rectangle, 2 by $\sqrt{2}$. The $\sqrt{3}$ might have been taken by constructing a right triangle with base 1 and hypotenuse 2. The altitude evidently being the $\sqrt{3}$. The $\sqrt{8}=2\sqrt{2}$; set compass $=\sqrt{2}$ and lay off length twice. Take for instance,

$$y=(3+2\sqrt{5})/7.$$

We would construct $\sqrt{5}$ as above. Set the compass to this length and step it off twice in the required direction, then set to unit length and step off 3 in addition. The ordinary geometrical method of dividing a line into equal parts will now suffice to get the seventh part, but it will be just as accurate for our purpose to obtain it by trial with the dividers as the ordinary draughtsman does. The necessity of rationalizing the denominator is evident both in the numerical process and the geometric counterpart.

For extensions of the approximation process to other forms of computation abbreviated methods of computation with exact numbers such as the Italian method of division and the like, I will refer the interested teacher to such books as Langley's "Computation," Longman, Green & Co., New York; Skinner's "Approximate Computation," Henry Holt & Co., New York; De Morgan's "Elements of Arithmetic" (this last probably obtainable only in libraries); or any good English arithmetic. *I would like to emphasize the fact that I referred to the interested teacher, keeping in mind that the teacher's standpoint is different from the pupils and that there are many things that the teacher can introduce, should introduce and must introduce at the psychological moment, without reference to any set course of study or supervision of principal or superintendent.* Our curriculum to-day is clogged with courses which were meant to be parts of several main subjects. Intense specialization and over zealousness to exploit some single idea for advertising or other purposes has led us many times to cut asunder things that have no life apart.

In ordinary simple measurements we have an opportunity to find examples of mathematical thought that will enable our boys and girls not only to realize that mathematics is not a mere juggle of symbols by the turning of a crank, but a science full of the very beauties our psychologists say it lacks.

The curious questioning mind of the child, contemplative as it is even in the cradle, will easily become what we call the inquiring mind—the mind of the investigator if it is not dwarfed by system in education. Meditation by teacher and pupil as well as supervisor is imperative. The simple, toilful inductive methods of the investigator are not often seen in the glittering array of logic of the finished product. To primitive instinct to conceal the treasure house and from a desire to present the broad view gained only after tedious search, added to a wish to save time we can trace the ordinary deductive form of presentation. This style is only possible after having a knowledge of the result. Can we not let the child and the ordinary layman into the secret and take away some of the trappings and illusions and teach more real power instead of so many cyclopedic statements. At least we can show him that the carrying out of mathematical operations beyond certain limits is illogical and useless and absurd and untruthful and that even as it is desirable for a speaker or writer to know when to stop—so even more so is it for the computer.

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TEACHERS' SALARIES AND COST OF LIVING.

By L. D. Arnett.

Within the last decade several important investigations have been made bearing on the question of teachers' salaries. As a result a very considerable amount of data has been collected which indicates the position the teacher occupies in the class of salaried and wage-earning people and also to some extent the status of the teaching profession. The facts thus brought together have served a good purpose, because they have in general directed attention to some of the obstacles in the way of a teaching profession, among which may be mentioned low salaries, insecurity of the teaching position, and the short annual school term.

In the light of these investigations I wish to consider the question of salaries and cost of living. The teaching body from the kindergarten or rural school up to the university includes practically as many classes of teachers as there are classes of schools. To whichever class the teacher may belong, he enjoys a high social position in the community and as a result is expected to maintain a standard of living in proportion to his position. In the higher ranks this position compares favorably with that of the doctor or lawyer. In the lower ranks it is higher than that of persons who receive an equal amount of salary in other pursuits. And therefore, a statement which I wish to make at the outset is that the relation, in general, between occupations and social position, and consequently standard of living, while it may vary to some extent for different occupations, does not hold true for the teaching profession.

As has been mentioned frequently in articles on the subject, there are several factors which influence salaries. Among these may be mentioned the number of available teachers in the local district, the professional training and experience of teachers, a minimum salary law, and finally the wealth and progressive spirit of the community.

These factors I shall not stop to discuss, but pass to some of the recent investigations of salaries in which they are considered.

I wish to speak first of one conducted within the past year by the Association of Collegiate Alumnæ. This organization is composed of about two thousand graduates of women's colleges and co-educational institutions. The members of the organization are located in different parts of the country, and the investigation is based upon returns from 460 members. They estimate that the lowest salary a college woman can be expected to live upon is $500. They find, however, that 193 out of 307 teachers receive an initial salary of $500 or less, that 66 receive as an initial salary $400, and 33 receive only $300. They also find that 71 of this number receive from $600 to $700, 22 receive $800, and 21 from $900 to $1,100. But by way of contrast they find that similar conditions exist in other occupations than teaching. Twenty-three out of forty-two college graduates commenced their occupations at a salary of $500 or less and ten received from $600 to $700.

They find that salaries increase with duration of service as follows: Of 335 teaching salaries considered, twenty-seven per cent. are from $700 to $900, and in order to reach this wage, eleven have been teaching two years; seven, three years; fourteen, from four to six years; sixteen, from six to eight years; ten, from eight to eleven years; six, from eleven to fifteen years; and two, over fifteen years. Of the fifty-three teachers earning from $900 to $1,100, twenty-seven have been working over six years and seventeen over eight years; and of the fifty-two earning $1,100 to $1,200, thirty-two have been working over ten years. In regard to the effect of experience on salaries, they conclude that the teacher must expect to await from ten to fifteen years in order to receive a salary of $1,200 or more.

A majority of those reporting expend from $200 to $250 a year for clothing and they estimate the lowest cost for board at from $6 to $9 a week. Less than one half the number spend fifty to one hundred dollars each year in travel. The study concludes with some recommendations, one of which is that the organization endeavor to direct women into other lines of work than teaching.

The Illinois Educational Commission in an investigation conducted within the last two years, found that the average salary of women employed in the rural schools of that state was $39.62 a month and that they received about $250 a

year. The average salaries of men in the rural schools was $47.47. They estimated that the cost of food which in 1907 averaged $374.75 per family, is somewhat less than one half the expenditure of the entire family. A salary of forty dollars per month, an amount paid in some counties of the state for seven and one half months in the year, is less than half enough to support a family. In regard to the effect of low salaries they conclude by saying: "Clearly, then, at the present rate of wages we must depend largely upon unmarried women for our supply of teachers. This may not mean poor teaching, but it does mean that so long as this is the case teaching cannot become a profession." The commission concludes its investigation by recommending the enactment of a minimum salary law.

A more exhaustive investigation was made by the Minnesota Educational Association in 1906. This investigation is based upon returns from 1,000 teachers and with the coöperation of the teachers, data was collected bearing on the question of wages in general. They found that the average annual salary for all Minnesota teachers was $411. In comparison with other occupations they found that the average annual teachers' salary was less than the annual wage received by the following classes of laborers: Farms laborers, railroad section men, painters, boiler-makers, carpenters, machinists, plasterers, plumbers, printers, steam-fitters, stone-masons, blacksmiths and stone cutters.

They also state, as a result of low salaries and the insecurity of the teaching position in Minnesota, that more than fifty-eight high school superintendents have left the teaching ranks within the last ten years. They mention as causes for low salaries, "lack of appreciation," "low standards, hence too many teachers," "no professional spirit," "teachers not sufficiently organized," "lack of funds," etc. The investigation concludes by recommending that hereafter the license to teach be restricted to those persons who have had at least one year of professional training, that no license be granted to persons under eighteen years of age, and that a law be enacted fixing a minimum salary.

Other investigations might be mentioned but enough data have been given to indicate that salaries are low and that the best educational results can not be secured under such conditions.

Teachers' salaries have however been steadily rising for several years. As a rule, they have been fixed with reference, more especially, to the standards admitting to the teaching ranks, and while there is no method of measurement they have in a fair degree kept pace with these. Within recent years the cost of living, as an additional factor in determining the amount of salary, has been demanding an increased amount of attention by boards. I wish to compare, briefly, the rate of increase of salaries with the rate of increase of cost of living.

From the latest statistics of the Bureau of Education, the average monthly salary of male teachers in the public schools of the United States for 1907–08 was $62.35, and for female teachers, $51.61, while for 1897–98, the average for male teachers was $45.16, and for female teachers, $38.74. This shows, as far as we have data on the question, that during the ten years preceding 1907–08, the average monthly salary of male teachers increased $17.19 or about 38 per cent., and that for the same period the average monthly salary of female teachers, increased $12.87 or about 33 per cent. From a recent bulletin of the Bureau of Labor we find that the retail prices of food increased 25.2 per cent. during the same period and that the wages of labor increased 29.3 per cent. These comparisons are based upon data for the entire country.

The Middlesex County Teachers' Association of Massachusetts in 1906, found that the "living expenses of the teachers in twenty-two towns and cities in the county had increased 19.3 per cent. during the past ten years, while the salaries of these teachers had increased 10.8 per cent. during that time."

Professor Adams, of the University of Nevada, in a report recently published, says that "Teachers' salaries in Nevada when considered in relation to the cost of living, have fallen about 30 per cent. below the 1896 level." That is to say, the cost of living has risen very much faster than teachers' salaries.

Superintendent Barr, of Stockton, California, in a recommendation to the board of education, estimated that salaries increased only $11.39 or less than one and one half per cent. during the period from 1887 to 1907 and that the cost of living increased from 25 to 30 per cent. The state superintendent of Wyoming in his biennial report for 1907–08 says that "salaries of teachers have not increased correspondingly with the

increased cost of living, nor in proportion to the increase in remuneration in other lines of work generally."

Comparisons and estimates for other localities might be mentioned. The data which I have given indicates that in some parts of the country the cost of living has increased faster than salaries, but for the country as a whole, for the period from 1897–1907, the average monthly salary of public school teachers increased faster than the cost of living.

It has been suggested that the total annual amounts afford the only correct standards for comparing wages and salaries with the cost of living. The per cent. of increase as shown by the reports of the Bureau of Education is based upon the monthly salary, and does not take into consideration the wide difference of length of the school term in different parts of the country and is therefore misleading to that extent.

At a salary of $50 a month, an increase of ten per cent. would add $25 to the annual salary for a term of five months, but would add $45 for a term of nine months. Hence when all the factors are considered the results will probably show that salaries have not increased as rapidly as the cost of living and as local data indicates that in many sections of the country they are below the teacher's estimate of the cost of living, according to the standard he is expected to maintain, and further, that they are too low for the best interests of the school.

This brings me to the question of measures of relief and I wish to consider only the more important suggestions and remedies mentioned in the investigations. A remedy suggested by the Collegiate Alumnæ is that the number of teachers be reduced by directing college graduates into other pursuits. This plan would lessen the competition, and if executed very extensively, would certainly affect salaries. Graduates of colleges are needed in public school work and a plan of this kind would hardly commend itself to superintendents and principals of public schools. A second remedy suggested is the minimum salary law. Such laws have been enacted in Indiana, Maryland, New Jersey, North Carolina, North Dakota, Ohio, Pennsylvania and West Virginia. All cities that have adopted salary schedules, which provide for an increase of salary based on experience, follow the same plan. The law fixes the minimum monthly annual salary and thus establishes a standard. It is

claimed that the general effect of such a measure is good in that it operates to increase salaries above the minimum.

A third remedy suggested is that teachers be paid for twelve months in the year. This plan of payment is followed in a few cities, viz., Hoboken, Jersey City, Oakland and San Francisco, and has been recommended by superintendents of other cities. A final suggestion is that of publicity of facts. If all the facts relating to low salaries, the rate of increase of salaries, and the rate of increase of the cost of living, are made known to the public, and certainly to the salary board much better results might be obtained. No harm can come from such a method and much good may follow. This plan as a means of arousing interest in behalf of teachers' salaries is to be commended.

In conclusion I wish to say that the question of a high or low salary rests very largely with the teaching body. In its broader sense it should concern not only the teacher but the entire community. The manufacturer knows the value of skilled workmen; the merchant, the value of superior salesmen; they admit that it is economy to pay a higher wage in order to retain this class of help. The public should realize that the same economic principle is true in the conduct of schools. Enlightenment in these matters as well as in all of those that cocnern the question of salaries must come from the teaching body. During the past year salaries were increased in a comparatively large number of cities and towns. Special legislation regarding teachers' pensions, affecting either certain cities, or the state as a whole, was enacted in California, Maryland, Minnesota, Nebraska, Pennsylvania, Rhode Island, Virginia and Wisconsin.

One state, New Jersey, enacted a fairly permanent tenure of office law. Centralization of rural schools is in progress in some parts of the country.

These movements are significant, and while they do not all contribute to the immediate welfare of the teacher, they do contribute to the advancement of the cause of a teaching profession, and indirectly to the cause of higher salaries.

Bureau of Education,
Washington, D. C.

A GENERALIZED DEFINITION OF LIMIT.*

By E. D. Roe, Jr.

The object of this paper is to suggest a logical point of view from which a generalization of the definition of limit may be secured.

After some discussion it is suggested to so phrase the definition that not only is the usual definition logically included as a special case, but "infinity" is also included as a "limit," as well as any complex number.

Most of the suggestions made in the paper have been yearly presented to his students in calculus by the writer beginning with and since the year 1893.

It is with considerable diffidence that he presents his views, yet at the same time he feels moved to do so.

First we have to note for the purposes of these suggestions that the word "number" may have two meanings: First as an "expressible number." In this sense a number is said to be expressible when its modulus satisfies the statement

$$m < \text{mod (number)} < n,$$

where m and n are positive commensurable numbers. That is, if a number in this sense is present, two commensurable values, m and n, can always be found lying on either side of its modulus, and hemming it in, so to speak.

In this sense

0 is not a number,

ϵ is a number (ϵ being indefinitely small, but $\epsilon \neq 0$),

$1/\epsilon = \omega$ (ω being indefinitely great, but $\omega \neq \infty$),

∞ is not a number ($\infty = 1/0$).

But there is a second sense in which we may think of number as any symbol other than a symbol of operation which is capable of occupying a place in a combination or formula composed of like symbols. In this sense 0 and ∞ are also numbers, and

* Presented at the Meeting of The American Mathematical Society, September 7, 1910.

we shall also desire to speak of them as numbers in this generalized sense in our suggestions as well as to distinguish them from all other numbers in the first sense of "expressible number."

Next we must note that the word "limit" is multiple valued. Thus we have "limit" of the roots of an equation, "limit" of integration, or of summation, as well as, "limit" of a variable. It is in this last sense however that we wish to consider the word. And we note here by considering the distinction of "expressible" number in the first sense, that in the "theory of limits" it is in fact already used in two senses, viz:

1. *As limit or negation of all number* that may be expressed as defined in the first sense. *This limit or negation of number is not and can not be a number in the first sense.* As such we already recognize, and use a symbol (0) for zero.

Is there not another limit or negation of expressible number? viz., 1/0, or that for which the word "infinity," and the symbol ∞, might be reserved? Thus may it not be said: 0 fails of expression because there is nothing to express. ∞ fails of expression because it is beyond expression. 0 is absolute nothing; is no magnitude. It is the limit of all indefinitely small magnitudes that may be expressed. ∞ is not nothing. It is magnitude which cannot be expressed; it is all magnitude that can exist or be thought to exist, and as such is the limit of all indefinitely great magnitudes which may be expressed. Thus 0 and ∞ are both limits of number in the first sense, for they are denials or negations of the expressibility of number, the one by denying magnitude, the other by affirming all magnitude and thus denying its expressibility, and both exist as limits placed there by thought. As there can be but one zero and one infinity, and even though they are not numbers in the first sense, they are both absolute constants of thought in their nature.

2. *As a fixed number, between which and a variable the absolute value of the difference may be and is made indefinitely small.* This limit is always a number in the first sense. It may be noted that mathematicians generally assign to zero the role of a fixed number, in this second sense of the use of the word "limit."

With the help of the unique geometrical representation of the variable in the complex plane, the following is suggested as a definition of limit, which shall not only include both these meanings as special cases, and which refer only to the real variable, but which shall include the complex variable as well, and thus both generalize and unify the treatment of the subject. In this definition zero and infinity are to be classed as numbers in the sense of the second meaning of number. The definition will include zero and infinity as limits.

DEFINITION.—*One number a is the limit of another variable number z when a relation exists between them such that under the law of its change no circle may be drawn around the first number inside of which the second may not pass and remain as it approaches the former, but without, however, absolutely and permanently reaching it.*

To elucidate how inclusive this definition is we note the following.

SPECIAL CASES.

1. If a represents a real finite constant number, we have the ordinary statement of limits satisfied, viz., that $a = Lx$, when

$$a - \alpha < x < a + \alpha, \quad \text{or} \quad \mathrm{mod}\,(x - a) < \epsilon.$$

That is our definition becomes exactly logically equivalent with the ordinary definition. Thus it is not in conflict with it, but includes it as a special case. *Zero is also included under a.* Fig. 1.

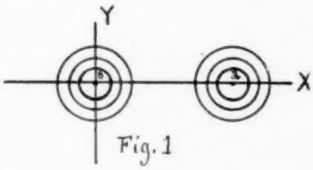

Fig. 1

2. If a represents a finite complex number, we see that our definition is logically equivalent with the following: *A finite complex number a is the limit of a variable number z when a relation exists between them such that under the law of its change the absolute value or modulus of the difference $z - a$ can be and is made to stay less than any assigned positive number ϵ however small without however becoming absolutely and permanently equal to zero.* Fig. 2.

3. If a represents *absolute infinity* the modulus of the difference cannot now be made indefinitely small. This then shows itself as a new case distinct from 1 and 2 but still included as a special case under the general definition. Fig. 3.

In case (3) the circles described about ∞ are parallel straight lines. The variable simply passes over them and goes off to an

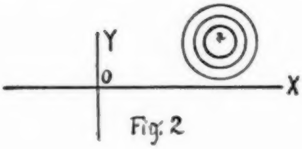

Fig. 2

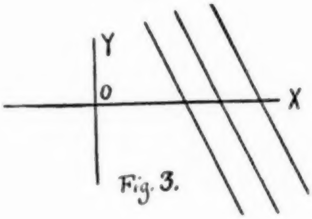

Fig. 3.

indefinitely great distance. No parallel however far away from O can be drawn over which the variable may not pass as it moves toward its limit ∞.

In case 1 we have $Lx = a$, including $L\epsilon = 0$.

In case 2 we have $Lz = a$.

In case 3 we have $L\omega = \infty$.

That is in case 3 we have a "limit" of the kind mentioned in (1), where we mean that ω is limited, not by any "expressible" number, it cannot be according to the definition, but only by the denial of the expressibility of number. To "a variable which increases without limit" ("limit" in the sense of (2), there is no "limit" of the kind mentioned in (2), no fixed and finite number, but there is a "limit" in the sense of 1, viz., a negation of expressibility, a negation that is required as a logical necessity of thought, an idea, or concept, just as zero is a negation, a logical necessity of thought, an idea or concept. We shall say then that 1/0 is a symbol representing magnitude, but not expressible number, indeed it is a symbol representing the negation of expressible magnitude; it is a superior limit of expressible number, a thought limit, not a numerical one, just as zero is a non-numerical inferior thought limit.

By adopting this generalized definition we have justified writing the statements:

$$\epsilon \doteq 0, \quad x \doteq a, \quad z \doteq a, \quad \omega \doteq \infty.$$

In conclusion the following observations on the definition may be made:

1. Some authors would have preferred to omit the words "but, without however, absolutely and permanently reaching it" from the definition. The writer however preferred to include them, for the sake as he considers it of not removing the props from under the foundation of a sound and logical theory of limits. On the other hand he is also glad to acknowledge that there are others who are not of his way of thinking.

A variable may pass through its limit in approaching it, but it is not permanently equal to it, as in the case of the variable $\sin x/e^x$ which passes through its limit zero infinitely many times as it approaches it, as $x \doteq \infty$.

"The limit of a constant is that constant" can always be eliminated by associating the constant with a variable by addition, subtraction, multiplication or division.

2. The definition is intended only for continuous function, or for such as whose representation by stereographic projection on the sphere would be continuous, or for the function on the side on which it is continuous (one-sided continuity). The continuous function is the only workable and usable function. It alone is subject to law and the laws of calculation. It is a loyal subject of the mathematical kingdom. Other so-called or miscalled functions are freaks, anarchists, disturbers of the peace, malformed curiosities which one and all are of no use to anyone, least of all to the loyal and burden-bearing subjects who by keeping the laws maintain the kingdom and make its advance possible. If such lawless freaks "reach their limits" or perform all sorts of acts impossible for the law abiding function, what does it signify? Least of all in order to legalize their acts shall the laws be done away with. It would seem that they should have reached their "limits" in another sense; and that scholarship lies in the direction of paying deference to the loyal continuous function rather than to the outlaws of mathematical society.

3. The definition will not be regarded with any favor by those who can not allow any vestige of a geometrical concept to run in their thoughts when they seek rigor. It is certain that the generalization cannot be made in terms of inequalities, though such will occur among the special cases. But the writer

(To be continued.)

believes it is an extreme view which would exclude every vestige of geometry from thought, because without criticism one may be misled by it. In the same way without sufficient care one may be equally misled by analysis. Error is not necessarily inherent in geometry or analysis, but in the thinking human mind.

4. By considering the special case (2), we see that $Lz = a$, may be written $L(x+yi) = \alpha + \beta i$, whence $Lx = \alpha$, $Ly = \beta$, which suggests another point of view, from which the generalization may be extended to the higher complex number. Thus if $Lx_1 = \alpha_1$, $Lx_2 = \alpha_2$, $\cdots$, $Lx_n = \alpha_n$, then $L(x_1 i_1 + x_2 i_2 + \cdots + x_n i_n) = \alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n$, in which the limit of each scalar is to be taken from the standpoint of the definition.

5. Finally the writer wishes to emphasize the fact that the suggestions made here constitute merely a point of view or one way of interpreting the facts. He has no quarrel with those who prefer to interpret them otherwise. He perfectly understands what is meant when one insists that we should write $x = \infty$, instead of $x \doteq \infty$. To him these signify exactly the same fact, only that the symbol ∞ is used with different meanings in the two statements. In the former it signifies a variable which is becoming indefinitely great. In the latter it signifies an absolute constant of thought, a negation of expressibility. In the whole presentation the writer has merely tried to explain that there is another way than that ordinarily proposed of looking at the facts involved in the theory of limits, and that instead of being in conflict with the ordinary view, it logically includes it as a special case.

Syracuse University,
12 September, 1910.

NEW BOOKS.

Shop Problems in Mathematics. By WILLIAM E. BRECKENRIDGE, SAMUEL F. MERSEREAU and CHARLES F. MOORE. Boston: Ginn & Company. Pp. 280. $1.00.

This book aims to give a thorough training in the mathematical operations that are useful in shop practice, *e. g.*, in carpentry, pattern-making, foundry work, forging and machine work, and, at the same time, to impart to the student much information in regard to shops and shop materials. The mathematical scope varies from addition of fractions to natural trigonometric functions. Problems are graded from simple work in board measure to the more difficult exercises of the machine shop.

All problems are based on actual experience.

The slide rule is treated at length. Short methods and checks are emphasized.

The book should be useful in any schools where there are shops: *i. e.*, in the upper grades of elementary schools for a review course in practical mathematics; in manual training high schools as a supplementary book of problems all through the mathematical course and in the shops; in trade schools as a textbook either in the mathematics classroom or in the shop; in normal schools; in apprentice schools; and in the classes of the Y. M. C. A.

Plane Geometry. Revised by GEORGE A. WENTWORTH and DAVID EUGENE SMITH. Boston: Ginn & Company. Pp. 287. $.80.

In this revision of Wentworth's "Geometry" every effort has been made not only to preserve but to improve upon the simplicity of treatment, the clearness of expression, and the symmetry of page that have characterized the successive editions of the book. The proofs have been given substantially in full and the sequence of propositions has been improved in several respects, notably in the treatment of parallels. The number of propositions has been decreased so as to include the great basal theorems and problems. The exercises have been rendered more dignified in appearance and have been improved in content. The number of simple exercises has been greatly increased. Definitions have been postponed until they are actually needed and only well-recognized terms have been employed.

NOTES AND NEWS

THE ANNUAL MEETING OF THE ASSOCIATION will be held on Saturday, November 26, at the University of Pennsylvania, Philadelphia, Pa. A splendid program is being prepared and it is hoped that a large number of members will attend.